Homology for Operator Algebras II : Stable Homology for Non-self-adjoint Algebras

S. C. Power

Department of Mathematics
Lancaster University
England LA1 4YF

Abstract

A new homology is defined for a non-self-adjoint operator algebra with a distinguished masa which is based upon cycles and boundaries associated with complexes of partial isometries in the stable algebra. Under natural hypotheses the zeroth order group coincides with the K_0 group of the generated C^*-algebra. Several identifications and applications are given, and in particular it is shown how stable homology is significant for the classification of regular subalgebras and regular limit algebras.

Non-self-adjoint operator algebras are usually given in terms of a construct from a more primitive category. Such categories include partially ordered measure spaces (for nest algebras and commutative subspace lattice algebras), semigroup actions (for semicrossed products), ordered Bratteli diagrams (for subalgebras of AF C^*-algebras), and binary relations and groupoids (for various subalgebras of coordinatised von Neumann algebras and C^*-algebras). Throughout the literature there has been a great emphasis placed on relating operator algebras to the pertinent aspects of their genesis in the simpler category. In the present paper we introduce various stable homology groups $H_n(\mathcal{A};\mathcal{C})$ for operator algebras $\mathcal{A}$ with a prescribed self-adjoint subalgebra $\mathcal{C}$. The case of a digraph algebra provides the root context and here stable homology is coincident with the integral simplicial homology of the simplicial complex of the underlying directed graph. Although intrinsically defined the stable homology groups, in contrast to those of Hochschild cohomology, are often instantly computable from the underlying construct. At the same time these groups are related significantly to the algebraic structure. On the other hand, it is immediate from the definition that stable homology provides isomorphism invariants for the most natural isomorphisms, namely those with C^*-algebra extensions.

Although the new homology groups are of interest in their own right, and in counterpoint with Hochschild cohomology, they acquire added significance with regard to the classification of so-called *regular* subalgebras of non-self-adjoint operator algebras. The results here are of interest even in the finite-dimensional case. All self-adjoint subalgebras of finite-dimensional C^*-algebras are regular and they are well-understood in terms of Bratteli diagrams. The non-self-adjoint generalisation of this is to understand regular subalgebras of digraph algebras, both in terms of generalised Bratteli diagrams, and in terms of induced maps on the K_0 and homology groups, together with other related invariants. This is necessary to describe not merely

the nature of subalgebras, but also their possible positions, that is their classification up to inner conjugacy. Even in the case of rather simple digraphs, such as the cube Cu of section 3, this raises some interesting combinatorial problems.

The structure and classification of regular subalgebras in finite dimensions is also a necessary prelude to the classification of limit algebras (even algebraic direct limits) in the style of Elliott's classification of AF C^*-algebras in terms of the scaled K_0 group. Such ideas have already appeared in [20] where it has been shown how certain limit homology groups arise in the analysis of limits of digraph algebras based on cycles. The homology group formulations below give an alternative more generally applicable approach to these limit groups.

The underlying idea for stable homology is simply the following. Self-adjoint projections can play the role of 0-simplices, and partial isometries can play the role of 1-simplices. The formulation should provide a zeroth order homology group that is coincident with the K_0 group for the generated C^*-algebra. (In the case of a digraph algebra this is a free abelian group whose rank is equal to the number of components of the digraph.) And the formulation should provide nonzero elements in the first homology group if there are (appropriate) cycles of partial isometries which are not expressible as boundaries in any larger supercomplex. In this fashion we can obtain a homology theory in which we can identify contributions from partial isometry cycles that are linked to specific elements of (the positive cone of) the K_0 group of the generated C^*-algebra. In brief, we define $H_n(\mathcal{A};\mathcal{C})$ in terms of the simplicial homology of certain cycles of $\mathcal{C}$-normalising partial isometries in the stable algebra of $\mathcal{A}$. Although we concentrate on operator algebras in the text this geometric form of homology is also applicable to subspaces of C^*-algebras which are bimodules for a distinguished self-adjoint subalgebra. In most of the examples we look at the elements of $H_n(\mathcal{A};\mathcal{C})$ are

already generated by partial isometries of $\mathcal{A}$, rather than partial isometries of the stable algebra. Nevertheless the stable algebra formulation seems to be appropriate for the purposes of classification and of providing computable higher order obstructions to the vanishing of Hochschild cohomology. If $\mathcal{A}$ itself is self-adjoint then these groups vanish for $n \geq 1$, *for entirely trivial reasons*. This already suggests that these invariants are particularly appropriate to general operator algebras and are intrinsically more computable than Hochschild cohomology.

The paper is organised in the following way. In the first section we define stable homology and remark on the inadequacies of some variants of this definition. In section 2 we identify stable homology in some fundamental settings (i), the tensor product of a digraph algebra and a general C^*-algebra (Theorem 2.1 provides an elementary Kunneth formula), (ii) non-self-adjoint subalgebras of factors determined by a finite lattice of commuting projections, and (iii) regular limits of digraph algebras. In the latter case we recover the homology limit groups introduced in Davidson and Power [2]. We also mention a connection between the first stable homology group and certain locally inner automorphisms.

In the remainder of the text we give three related applications. Section 3 is concerned entirely with finite-dimensional matters : regular subalgebras (and inclusions) of digraph algebras, rigid embeddings, and the $K_0 \oplus H_*$ uniqueness property, particularly in the context of cycles, suspensions, discrete tori, and the cube algebra (this being a higher dimensional variant of the 4-cycle algebra). In section 4 we indicate how such classifications may be extended to similar settings in AF C^*-algebras by considering *scaled* homology groups. In the final section we illustrate how homology can appear in the classification of regular limit algebras. It is clear that there are some very interesting classification problems in this area and we hope to develop these ideas more fully elsewhere. Note that the final two sections are independent

of the first two in the sense that one can consider the homology groups there as limit groups (cf. Theorem 2.5).

Let us remark very briefly on the current literature concerning the cohomology and homology of non-self-adjoint operator algebras.

Automorphisms and derivations have formed a central topic in operator algebra - one which is closely connected to the more general considerations of Hochschild cohomology. In the realm of reflexive algebras the vanishing of Hochschild cohomolgy for nest algebras has been demonstrated by Lance [15] and Christensen [1], whilst nonzero cohomology and non-inner derivations have been identified and studied by Gilfeather [4], Gilfeather, Hopenwasser and Larson [5], Gilfeather and Moore [6], and Power [24].

Traditional studies of Hochschild cohomology for function algebras, as propounded by Helemskii [10], Johnson [11] and Taylor [26], for example, have direct bearing on operator algebras in the abelian case. However a number of more recent studies have been pointed specifically towards non-commutative algebras. In particular Gilfeather and Smith [7], [8] and [9] have examined Hochschild cohomology for constructions of operator algebras analogous to the join, cone and suspension constructions that are available in simplicial homology. This work was inspired partly by the cohomological identifications of Gerstenhaber and Schack [3] and Kraus and Schack [14] who promoted the fact that (for digraph algebras) Hochschild cohomology is identifiable with a simplicial cohomology. The analysis of [7], [8] and [9] also leans on basic techniques of Johnson, Kadison and Ringrose ([12], [13]) in the Hochschild cohomology theory for C^*-algebras and von Neumann algebras.

In a different direction, but also motivated by digraph algebras, Davidson and Power [2] considered direct limit homology groups for triangular limit algebras, and in [20] it was shown that these could be used as classifying invariants in certain contexts of 4-cycle limit algebras. This homology theory,

like that which we have given for reflexive algebras in [24], is closely tied to the underlying coordinatisation of the algebra, and is possibly more appropriate and computable than Banach algebra cohomology. The present paper gives a general intrinsic formulation for these groups which is quite widely applicable. We envisage that these invariants will be accessible and significant in the area of subalgebras of groupoid C^*-algebras, as developed by Muhly and Solel [16] in the triangular case, and in the (completely undeveloped) area of direct limits of non-self-adjoint subhomogeneous algebras.

The following terminology is adopted. A digraph algebra $\mathcal{A}$ is a subalgebra of a complex matrix algebra M_n which contains a maximal abelian self-adjoint subalgebra (a masa). These are also known as finite-dimensional CSL algebras or finite-dimensional incidence algebras. If $\{e_{i,j}\}$ is a standard matrix unit system for M_n such that the masa in question is spanned by the matrix units $\{e_{i,i}\}$ then the digraph for $\mathcal{A}$ has n vertices and directed edges (i,j) for each $e_{i,j}$ in $\mathcal{A}$. This digraph (or binary relation) is transitive and reflexive, with no multiple directed edges. From the point of view of cohomology and homology the digraph algebras $A(D_{2n})$ for the $2n$-cycle digraphs D_{2n} are the first significant examples. These algebra are also denoted $\mathcal{A}_{2n}$ and are occasionally refered to as the (finite-dimensional) tridiagonal matrix algebras. All the algebras that we consider are viewed as subalgebras of C^*-algebras, and by a star-extendible homomorphism we mean one which is a restriction of a C^*-algebra homomorphism between the generated C^*-algebras.

1 Formulations of Stable Homology

Let $\mathcal{A}$ be an operator algebra with self-adjoint subalgebra $\mathcal{C}$. Our main interest is when $\mathcal{C}$ is maximal abelian. In the following discussion it should be held in mind that we seek to formulate a stable homology theory, in the sense that $H_n(\mathcal{A}\otimes M_n;\mathcal{C}\otimes\mathbb{C}^n) = H_n(\mathcal{A};\mathcal{C})$, we wish to have $H_0(\mathcal{A};\mathcal{C}) = K_0(C^*(A)$) in appropriate contexts, and we require that $H_*(\mathcal{A};\mathcal{C})$ specialises to integral simplicial homology in the case of digraph algebras. Furthermore, we wish to have the elementary Kunneth formula of Theorem 2.1 which links K_0 and H_*.

An alternative elementary formulation of $H_1(\mathcal{A};\mathcal{C})$, which is independent of simplicial homology, is given in Remark 1.3.

The stable algebra of an operator algebra $\mathcal{A}$ is taken to be the algebra of finitely nonzero infinite matrices over $\mathcal{A}$. Let B be a finite-dimensional C^* -algebra contained in the stable algebra $M_\infty(C^*(\mathcal{A}))$ with a full matrix unit system $\{f_{ij}\}$ consisting of partial isometries which normalise the subalgebra $M_\infty(\mathcal{C})$. This means that if $f \in \{f_{i,j}\}$ and $c \in M_\infty(\mathcal{C})$ then fcf^* and f^*cf belong to $M_\infty(\mathcal{C})$. Then the subalgebra

$$A = B \cap M_\infty(\mathcal{A})$$

contains the diagonal matrix units and so A is a subalgebra of B associated with the binary relation $R(A) = \{(i,j) : f_{i,j} \in A\}$. In particular A is completely isometrically isomorphic to the digraph algebra associated with $R(A)$. If A and A' are two such subalgebras of $M_\infty(\mathcal{A})$ then we declare them to be equivalent if they are conjugate by means of a unitary operator u in (the unitisation of) $M_\infty(\mathcal{A} \cap \mathcal{A}^*)$.

Let [A] denote the equivalence class of such digraph subalgebras, and let $H_n([A])$ denote the n^{th} integral simplical homology group of the simplicial

complex $\Delta([A])$ associated with $R(A)$. The complex $\Delta([A])$ is perhaps most easily specified by viewing $R(A)$ as the edges of a directed graph G with vertices $v_1, ..., v_n$: Let $\overline{G}$ be the undirected graph of G. Then the 0-simplices of $\Delta(G)$, denoted $\sigma_i = < v_i >$, $1 \leq i \leq n$, are associated with the vertices v_i of G, and the t-simplices of $\Delta(G)$ correspond to the complete subgraphs of $\overline{G}$ with $t+1$ vertices. Thus if v_i, v_j, v_k determine a complete subgraph of $\overline{G}$ then the 2-simplex $\sigma_{ijk} =< v_i, v_j, v_k >$ is included in $\Delta(G)$.

The group $H_n(\mathcal{A};\mathcal{C})$ is defined to be the quotient

$$H_n(A;\mathcal{C}) = (\sum_{[A]} \oplus H_n([A]))/Q_n$$

where the direct sum indicates the restricted direct sum, and where Q_n is a natural subgroup corresponding to inclusion identifications and to identifications arising from certain orthogonal direct sums (induced decompositions) as described below. Roughly speaking, it follows that $H_1(\mathcal{A};\mathcal{C})$ is nonzero if there exists a sequence of normalising partial isometries in $M_\infty(\mathcal{A})$ which form a 1-cycle in a finite-dimensional algebra A but which do not give a 1-boundary in any affiliated algebra A' containing A.

We now define Q_n. Refer to the algebras A, A', as above, as $M_\infty(\mathcal{C})$-*normalising* (or $\mathcal{C}$-*normalising*) *digraph algebras for* $\mathcal{A}$, and refer to the matrix unit system $\{f_{i,j} : f_{i,j} \in A\}$ as a *partial matrix unit system for* A. Note that such a system has the special property that the generated star semigroup is a full matrix unit system in the usual sense. Let $A \subseteq A'$ be $\mathcal{C}$ -normalising digraph algebras such that the partial matrix unit system of A is a subset of the partial matrix unit system for A'. Then there is a natural well-defined group homomorphism $\theta : H_n([A]) \to H_n([A'])$ which is induced by the resulting digraph **inclusion** $R(A) \to R(A')$. Identify each group $H_n([A])$ with its summand in $\sum_{[A]} \oplus H_n([A])$ and let Q_n^a be the

set of elements of the form $g - \theta(g)$ associated with all such group homomorphisms $\theta : H_n([A]) \to H_n([A'])$, and elements g in $H_n([A])$. Of course there may be a finite number of such group homomorphisms for each pair $[A], [A']$. Note, in particular, that we only consider rather special inclusions which, in the terminology of section 3, are multiplicity one regular inclusions.

The subgroup Q_n is defined to be the subgroup generated by Q_n^a and Q_n^b, where Q_n^b corresponds to certain orthogonal direct sum identifications, as we now indicate.

Let A be a $\mathcal{C}$-normalising digraph algebra for $\mathcal{A}$ with partial matrix unit system $\{f_{i,j} : (i,j) \in R(A)\}$. Without loss of generality assume that $C^*(A) = M_n$. Let $f_{11} = f'_{11} \oplus f'_{22}$, with f'_{11}, f'_{22} nonzero projections in $M_\infty(\mathcal{C})$. Then, since the $f_{i,j}$ are $\mathcal{C}$-normalising, it follows that there is an **induced decomposition** $f_{ij} = f'_{ij} + f''_{ij}$, for (i,j) in $R(A)$, such that

$$\{f'_{ij} : (i,j) \in R(A)\} \quad \text{and} \quad \{f''_{ij} : (i,j) \in R(A)\}$$

are partial matrix unit systems for $\mathcal{C}$-normalising digraph algebras A', A'' respectively. In fact $f'_{ij} = f_{i,1} f'_{1,1} f_{1,j}$. Let $\theta' : A \to A'$, $\theta'' : A \to A''$ be the associated algebra isomorphisms, with induced (well-defined) isomomorphisms

$$\theta'_n : H_n([A]) \to H_n([A']), \quad \theta''_n : H_n([A]) \to H_n([A'']).$$

Define Q_n^b to be the set of elements of $\sum_{[A]} \oplus H_n([A])$ of the form

$$g - \theta'_n(g) - \theta''_n(g) \quad , g \in H_n([A]).$$

The definition of the stable homology groups is now complete.

Definition 1.1 The $\mathcal{C}$ -*normal stable homology* of the operator algebra $\mathcal{A}$ with distinguished self-adjoint subalgebra $\mathcal{C}$ consists of the groups $H_n(\mathcal{A};\mathcal{C})$, $n = 0, 1, 2, \ldots$.

The discussion above gives a fairly intuitive construction and we shall see in the next section that it is quite suited to calculations in specific contexts.

Let us point out how the Grothendieck group $G(S)$ of an abelian unital semigroup S can be viewed in the above formalism. Let

$$\mathcal{G} = (\sum_{s\in S} \oplus \mathbb{Z})/\mathcal{R},$$

where $\mathcal{R}$ is the subgroup generated by the elements associated with the relations for the semigroup S. (A typical such element has the form $n_{s+t} - (n_s \oplus n_t)$ with s, t in S.) Then $\mathcal{G}$ is naturally isomorphic to the usual Grothendieck group of S. From this and our definitions above it follows that if $\mathcal{B}$ is a unital C^*-algebra then $H_0(\mathcal{B};\mathcal{B}) = K_0(\mathcal{B})$.

Similarly, let $\mathcal{C}$ be a unital C^*-subalgebra of $\mathcal{B}$ with the following properties : (i) for each projection class $[p]$ in $M_\infty(\mathcal{B})$ there is a projection q in $M_\infty(\mathcal{C})$ with $[p] = [q]$, and (ii) if q_1 and q_2 are projections in $M_\infty(\mathcal{C})$ which are equivalent in $M_\infty(B)$ then they are equivalent by an $M_\infty(\mathcal{C})$-normalising partial isometry. The first property implies that the natural map $K_0(\mathcal{C}) \to K_0(\mathcal{B})$ is a surjection. If (i) and (ii) both hold we shall say that the map $K_0(\mathcal{C}) \to K_0(\mathcal{B})$ is a *regular surjection*. Under these circumstances it follows that $H_0(\mathcal{B};\mathcal{C}) = K_0(\mathcal{B})$.

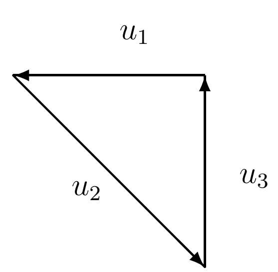
Remark 1.2 One can also present the homology groups $H_n(\mathcal{A};\mathcal{C})$ in a more orthodox fashion as the homology groups of a chain complex $(S_n(\mathcal{A}), d_n)$. To do this define $S_n(\mathcal{A})$ to be the quotient

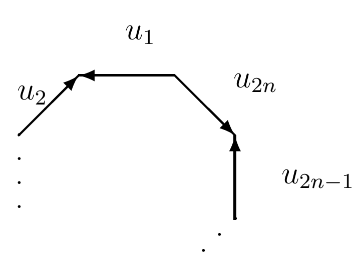
$$(\sum_{[A]} \oplus S_n([A]))/QS_n$$

where $S_n([A])$ is the n -chain group of the complex for $R(A)$, with integral coefficients, and where QS_n is the subgroup determined by the relations of inclusion of matrix unit systems and of orthogonal direct sum. The boundary operators d_n respect the subgroups QS_n and so we may define $Z_n(\mathcal{A})$, the n -cycle group, and $B_n(\mathcal{A})$, the n -boundary group. Then the quotient groups $Z_n(\mathcal{A})/B_n(\mathcal{A})$ are the homology groups of the associated chain complex $(S_n(\mathcal{A}), d_n)$ and they are identifiable with the groups $H_n(\mathcal{A}, \mathcal{C})$.

Remark 1.3 An alternative direct formulation of $H_1(\mathcal{A}; \mathcal{C})$ can be made in the following fashion.

A *basic* $M_\infty(\mathcal{C})$*-normalising 1-cycle* of $\mathcal{A}$ is a triple $\sigma = (u_1, u_2, u_3)$, or a $2n$ -tuple $\sigma = (u_1, \ldots, u_{2n})$, consisting of partial isometries in $M_\infty(\mathcal{A})$ which normalise $M_\infty(\mathcal{C})$ and satisfy the relations suggested by the following diagrams.





Thus, for the $2n$ -cycle, $d(u_{2k}) = d(u_{2k+1})$ and $r(u_{2k+1}) = r(u_{2k+2})$ for all appropriate k, and all these domain and range projections are orthogonal.

Furthermore,

$$u_{2n} = u_{2n-1}u_{2n-2}^* \dots u_2^* u_1,$$

and so $C^*(\{u_1, \dots, u_{2n}\})$ is isomorphic to M_{2n}, and the nonzero words in the elements $u_1, \dots, u_{2n}$ and their adjoints provide a complete matrix unit system for the algebra.

Let $[\sigma]$ denote the set of basic $M_\infty(\mathcal{C})$ -normalising $1-$ cycles ω that are unitarily equivalent to σ in the sense that $\omega = z\sigma z^*$ for some unitary z in (the unitisation of) $M_\infty(\mathcal{A}) \cap M_\infty(\mathcal{A})^*$. Define $Z_1(\mathcal{A};\mathcal{C})$ to be the free abelian group generated by such classes, modulo the following three relations:

(i) (orthogonal sum)

$$[(u_1, \dots, u_{2n})] + [(v_1, \dots, v_{2n})] = [(u_1 + v_1, \dots, u_{2n} + v_{2n})],$$

where representatives are chosen so that $u_i + v_i$ is a partial isometry for all i.

(ii) (cancellation)

$$[(u_1, \dots, u_{2n})] + [(u_{2n}, \dots, u_1)] = 0.$$

(iii) (addition) If $\sigma_1 = (u_1, \dots, u_{2n})$, $\sigma_2 = (v_1, \dots, v_{2m})$, $u_{2n} = v_1$, and $\sigma = (u_1, \dots, u_{2n-1}, v_2, \dots, v_{2m})$, then

$$[\sigma] = [\sigma_1] + [\sigma_2].$$

Define $B_1(\mathcal{A};\mathcal{C})$ to be the subgroup generated by the classes of the 1-cycles σ coming from triples. Then $H_1(\mathcal{A};\mathcal{C}) = Z_1(\mathcal{A};\mathcal{C})/B_1(\mathcal{A};\mathcal{C})$.

Remark 1.4 It is tempting to drop the normalising condition in the above formulations and define a stable homology in terms of all digraph subalgebras

of $M_\infty(\mathcal{A}\bigcap\mathcal{A}^*)$ with respect to unitary equivalence from $M_\infty(\mathcal{A}\bigcap\mathcal{A}^*)$. But this move leads to unwanted complications in view of the proliferation of unitary equivalence classes of partial isometry cycles, even when $\mathcal{A}$ is a digraph algebra. In fact one does not obtain a homology theory which generalises simplicial homology in this case. For example, in the case of the basic digraph algebra $\mathcal{A} = A(D_4)$ the resulting H_1 group is the restricted direct product of uncountably many copies of $\mathbb{Z}$. This is related to the fact that there are uncountably many inner equivalence classes of partial isometries in this algebra.

Remark 1.5 Here are three variations of stable homology:

(i) One could be more restrictive in the choice of partial matrix units by demanding that they normalise the diagonal algebra $D_\infty(C)$ rather than $M_\infty(C)$. This homology is somewhat more computable and it is adequate for the approximately finite settings considered in section 3, 4 and 5. However, there is the big disadvantage that one does not obtain a variant of Theorem 2.1 below.

(ii) One could drop the dependence on $\mathcal{C}$ altogether and define the homology groups $H_n(\mathcal{A};\mathcal{A}\cap\mathcal{A}^*)$. Here one requires normalisation of $M_\infty(\mathcal{A}\cap\mathcal{A}^*)$. This is a very attractive move, superficially, since the resulting groups are invariants for star-extendible isomorphism. Furthermore this homology does coincide with the simplicial homology of the digraphs of the digraph algebras. (See Theorem 2.2 (i).) However, the functoriality properties are seriously inadequate in the sense that regular morphisms between digraph algebras (such as the rigid embeddings in section 3) do not induce homology group homomorphisms. Furthermore in many basic contexts of interest these homology groups are clearly inappropriate. To see this consider the following example.

Ler $\mathcal{F}$ be the direct limit algebra $\varinjlim(M_{2^k},\phi_k)$ (not necessarily closed)

where $\phi_k(a) = a \oplus a$ for all a. Let $\mathcal{A}$ be the subalgebra of $A(D_4) \otimes \mathcal{F}$ consisting of the operators a for which $(e_{1,1} \otimes 1)a(e_{1,1} \otimes 1)$ belongs to $e_{1,1} \otimes \mathcal{D}$ where $\mathcal{D} = \varinjlim(D_{2^k}, \phi_k)$, the standard diagonal subalgebra. For the natural masa $\mathcal{C} = \mathbb{C}^4 \otimes \mathcal{D}$ the normal stable homology $H_1(\mathcal{A};\mathcal{C})$ is nontrivial and can be readily identified using Theorem 2.5. On the other hand $H_1(\mathcal{A};\mathcal{A} \cap \mathcal{A}^*)$ is trivial. This is essentially because the normalising demand is too great; if v is a partial isometry which normalises $\mathcal{A} \cap \mathcal{A}^*$ then $(e_{1,1} \otimes 1)v(e_{3,3} \otimes 1) = (e_{1,1} \otimes 1)v(e_{4,4} \otimes 1) = 0$.

(iii) One could *restrict* the class of partial isometries that are admisssible in the partial matrix unit systems of the digraph subalgebras. For example, in the operator algebra of Example 2.3 restriction to finite rank matrix units leads to a trivial first restricted stable homology group, and this reflects the triviality of the first simplicial homology group of the associated digraph of that example. This type of restriction seems appropriate for an analysis of the homology affiliated to elements of $K_0(C^*(\mathcal{A}))$.

Remark 1.6 The stable homology that we have given is defined in terms of finite-dimensional C^*-subalgebras. Even in the case triangular limit algebras with an "approximately finite-dimensional character" such an "AF homology" may be inappropriate. We have in mind here the limits of cycle algebras under *non-star-extendible* embeddings, given in [19] and [20]. It can be shown that these have trivial first stable homology (with resect to the unique masa). On the other hand they do posses natural nonzero limit homology groups (see [19]).

Remark 1.7 Minor modifications of the definitions above lead to the formulation of the relative homology groups :

Let $\mathcal{C}, \mathcal{A}$ be before, and let $\mathcal{A}'$ be an intermediate operator algebra with

$\mathcal{C} \subseteq \mathcal{A}' \subseteq \mathcal{A}$. Let B, A be as before, and let $A' = B \cap M_\infty(\mathcal{A}')$, so that A' is a $\mathcal{C}$-normalising digraph subalgebra of $\mathcal{A}'$ which is spanned by some of the matrix units of A. To the unitary equivalence class $[A, A']$ of such pairs associate the relative integral simplicial homology group $H_n([A, A'])$, which is defined to be the relative homology group $H_n(\Delta(A), \Delta(A'))$, where $\Delta(A')$ is the subcomplex determined by $R(A')$. Define the relative $\mathcal{C}$-normalising homology to be the quotient

$$H_n(\mathcal{A}, \mathcal{A}'; \mathcal{C}) = (\sum_{[A,A']} \oplus H_n([A, A']))/Q_n(\mathcal{A}, \mathcal{A}')$$

where $Q_n(\mathcal{A}, \mathcal{A}')$ is the subgroup of the restricted direct sum determined by orthogonal direct sum identifications, and by subcomplex identifications.

Alternatively, we can view the chain complex $(S_n(\mathcal{A}'), d_n)$ as a subcomplex of the chain complex $(S_n(\mathcal{A}), d_n)$, in which case $H_n(\mathcal{A}, \mathcal{A}'; \mathcal{C})$ is the homology of the quotient chain complex $(S_n(\mathcal{A})/S_n(\mathcal{A}'), d_n)$.

Remark 1.8 Stable homology is, prima facie, an invariant for pairs $(\mathcal{A}, \mathcal{C})$. However, in the presence of uniqueness theorems (up to automorphisms of $\mathcal{A}$) for regular masas $\mathcal{C}$, one can simply define $H_*(\mathcal{A}) = H_*(\mathcal{A}; \mathcal{C})$ and obtain a well-defined homology theory for $\mathcal{A}$ itself. Examples of this appear in sections 3 and 4 and we expect similar definitions of $H_*(\mathcal{A})$ in much more general circumstances. Of course, in the extreme case of triangular algebras, such as the lexicographic products in Example 2.6, the masa $\mathcal{C} = \mathcal{A} \cap \mathcal{A}^*$ is intrinsic to the algebra and we may define $H_*(\mathcal{A}) = H_*(\mathcal{A}; \mathcal{C})$.

2 Identifications of Stable Homology

We have remarked in the introduction that the stable homology of a digraph algebra coincides with the simplicial homology of the complex for the digraph of the algebra. The next two theorems establish this and give different more general versions of this correspondence. The proofs are essentially elementary and depend on the decomposition of an arbitrary $M_\infty(\mathcal{C})$-normalising digraph algebra in the stable algebra into an "parallel sum" of ones that are unitarily equivalent to certain easily visible elementary digraph subalgebras.

Theorem 2.1 *Let $A(G)$ be a digraph algebra and let $\mathcal{B}$ be a unital C^*-algebra with abelian unital self-adjoint subalgebra C such that the inclusion $C \to \mathcal{B}$ induces a regular surjection $K_0C \to K_0\mathcal{B}$. Then, for each $n \geq 0$,*

$$H_n((A(G) \otimes \mathcal{B}; \mathbb{C}^{|G|} \otimes C)) \;=\; H_n(\Delta(G)) \otimes_{\mathbb{Z}} K_0(\mathcal{B}).$$

Proof: Let $\mathcal{A} = A(G) \otimes \mathcal{B}$, $\mathcal{C} = \mathbb{C}^{|G|} \otimes C$. Here $\mathbb{C}^{|G|}$ is the diagonal subalgebra of $A(G)$ with respect to a fixed matrix unit system $\{e_{i,j} : (i,j) \in E(G)\}$. We may assume that G is connected. The main step is to reduce the quotient expression for $H_n(\mathcal{A}; \mathcal{C})$ to one involving a direct sum over standard type digraph subalgebras of the form $A(G) \otimes q$ where q is a projection in $M_N(C)$.

Let $A \subseteq M_N(A(G) \otimes \mathcal{B}) = A(G) \otimes M_N(\mathcal{B})$ be a digraph subalgebra with partial matrix unit system $\{f_{k,l}\}$ each element of which normalises $M_N(\mathcal{C}) = \mathbb{C}^{|G|} \otimes M_N(C)$. Without loss of generality assume that the digraph for A is connected and that the full system of $\{f_{k,l}\}$ is $\{f_{k,l} : 1 \leq k, l \leq K\}$.

Note the following principle : if a 2×2 operator matrix v is a partial isometry, say

$$v = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

and if vxv^* is block diagonal when x is

$$\begin{bmatrix} I_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix},$$

then a, b, c, d are partial isometries with orthogonal domains and ranges. Using this principle repeatedly obtain an induced decomposition $f_{k,l} = f_{k,l}^{(1)} + \ldots + f_{k,l}^{(t)}$, in the sense given in section 1, such that each $f_{k,l}^{(r)}$ has the normalising property and belongs to one of the spaces $e_{i,j} \otimes M_N(\mathcal{B})$. More explicitly, consider the projections $e_i = e_{i,i} \otimes 1$, $1 \leq i \leq |G|$, in $A(G) \otimes M_N(C)$. Then there is an induced decomposition $f_{k,l} = f'_{k,l} + f''_{k,l}$, where, for each pair k, l, $f'_{k,l} = f_{k,1}(f_{1,1}e_1)f_{1,l}$. If this is a nontrivial decomposition, that is, if $f_{1,1}e_1 \neq 0$, then $f'_{1,1}e_1 = f'_{1,1}$. Furthermore, the systems $\{f'_{k,l}\}$ and $\{f''_{k,l}\}$ still have the normalising property. Repeating such decompositions leads to the desired reduction.

For fixed r consider the associated *full* matrix unit system $\{f_{k,l}^{(r)}\}$. Then for each i the intersection

$$\{f_{k,l}^{(r)}\} \cap (e_{i,i} \otimes M_N(B))$$

is a complete system of matrix units and so has the form

$$e_{i,i} \otimes g_{s,t}^i \quad ,\text{for} \quad 1 \leq s, t \leq n_i,$$

where $\{g_{s,t}^i\}$ is a full matrix unit system in $M_N(\mathcal{B})$ which normalises $M_N(C)$. Let m be the maximum of the numbers n_i, say $m = n_p$. Since $f_{k,l}^{(r)}$ is a full matrix unit system the matrix unit $g_{s,s}^i$ is equivalent to $g_{s,s}^p$ for each s with $1 \leq s \leq n_i$, by a matrix unit of the form

$$f_{k,l}^{(r)} = e_{i,p} \otimes v$$

where v normalises $M_N(\mathcal{C})$. It follows that by conjugating we may assume that $g^i_{s,s} = g^p_{s,s}$ for $1 \le s \le n_i$. We now see that the full matrix unit system $\{f^{(r)}_{k,l}\}$ is conjugate, by a normalising unitary in $M_N(\mathcal{A} \cap \mathcal{A}^*)$, to a subsystem of a system of the form

$$\{e_{i,j} \otimes g_{s,t}\}$$

where $\{g_{s,t}\}$ is a complete matrix unit system for M_m as a (not necessarily unital) subalgebra of $M_N(\mathcal{B})$, with the normalising property.

To recap, it has been shown that A is inner equivalent, by a unitary in $M_N(\mathcal{A} \cap \mathcal{A}^*)$, to a digraph algebra with partial matrix unit system $\{f_{k,l}\}$ admitting an induced decomposition $f_{k,l} = f^{(1)}_{k,l} + \ldots + f^{(t)}_{k,l}$, where each partial system $\{ f^{(r)}_{k,l}\}$ is a subsystem of the standard system for an elementary digraph algebra $A(G) \otimes M_m \otimes q$, where q is a projection in $M_N(\mathcal{C})$, and m, N and q depend on r. In brief, each digraph algebra class $[A]$ for $\mathcal{A}$ has a representative digraph algebra which is constructed in a natural way from elementary ones.

Let

$$\mathcal{G} = \sum_{[A]} \oplus H_n([A]), \quad \mathcal{G}_0 = \sum_{[q],m} \oplus H_n([A(G) \otimes M_m \otimes q]).$$

where $\mathcal{G}_0$ is the subgroup of $\mathcal{G}$ associated with the elementary digraph subalgebras indexed by the $K_0(\mathcal{B})$ classes $[q]$, with q in $M_\infty(C)$, and positive integers m. Thus $H_n(\mathcal{A};\mathcal{C}) = \mathcal{G}/Q_n$, and, by the reductions above, $\mathcal{G}/Q_n = \mathcal{G}_0/Q_n$. Furthermore, $\mathcal{G}_0/Q_n = \mathcal{G}_0/Q_{n,0}$ where $Q_{n,0}$ is the subgroup generated by the set of relations $Q^a_{n,0}$, $Q^b_{n,0}$ corresponding to inclusions and induced decompositions for elementary digraph algebras. This is purely algebraic fact which follows from the simple principle that for abelian groups G, H the quotient group $(G \oplus G \oplus H)/\{g \oplus -g \oplus 0\}$ is isomorphic to $0 \oplus G \oplus H$.

The inclusion $A(G) \otimes e_{1,1} \to A(G) \otimes M_n$ induces an isomorphism of simplicial homology leading to the further reduction

$$H_n(\mathcal{A};\mathcal{C}) = (\sum_{[q]} \oplus H_n([A(G) \otimes q]))/Q^b_{n,0}$$

where the direct sum extends over classes of projections q in $M_\infty(\mathcal{C})$. (There are no remaining inclusion relations.) Thus, making the natural identifications $H_n([A(G) \otimes q]) = H_n(\Delta(G))$, we see that

$$H_n(\mathcal{A};\mathcal{C}) = (\sum_{[q]} \oplus H_n(\Delta(G)))/S$$

where S is the subgroup corresponding to the semigroup relations for the classes $[q]$. Hence

$$(\sum_{[q]} \oplus H_n(\Delta(G)))/S = H_n(\Delta(G)) \otimes_{\mathbb{Z}} ((\sum_{[q]} \oplus \mathbb{Z})/S).$$

Since the map $K_0 C \to K_0\mathcal{B}$ is a regular inclusion it follows that

$$K_0\mathcal{B} = (\sum_{[q]} \oplus \mathbb{Z})/S$$

and the proof is complete. $\square$

The next identifications are similar to the last but are somewhat more elementary.

Let $\mathcal{M}$ be a factor and let $\mathcal{L}$ be a finite lattice of commuting projections in $\mathcal{M}$ with associated subalgebra $\mathcal{A}$ consisting of the operators a in $\mathcal{M}$ for which $(1-p)ap = 0$ for all p in $\mathcal{L}$. The minimal nonzero interval projections $f - e$, with $f > e$ projections of $\mathcal{L}$, form a finite set, $Q = \{q_1, \ldots, q_n\}$ say. Q carries the transitive partial order $\ll$ where

$$q \ll q' \quad \Leftrightarrow \quad q\mathcal{A}q' = q\mathcal{M}q'.$$

Write $H_n(\Delta(\mathcal{L}))$ for the integral simplicial homology of the complex $\Delta(\mathcal{L})$ for the partial order $\ll$, viewed as a digraph.

Theorem 2.2 *Let $\mathcal{M}$ be II $_1$ factor and let $\mathcal{L} \subseteq \mathcal{M}$ be a finite lattice of commuting projections with associated reflexive algebra $\mathcal{A} \subseteq \mathcal{M}$. Then*

(i)

$$H_n(\mathcal{A}; \mathcal{A} \cap \mathcal{A}^*) = H_n(\Delta(\mathcal{L})) \otimes_{\mathbb{Z}} \mathbb{R}.$$

(ii) If $\mathcal{C} \subseteq \mathcal{M}$ is a regular masa of $\mathcal{M}$ then

$$H_n(\mathcal{A}; \mathcal{C}) = H_n(\Delta(\mathcal{L})) \otimes_{\mathbb{Z}} \mathbb{R}.$$

Proof: (i) Let A be a digraph algebra for $\mathcal{A}$ which is contained in $M_N(\mathcal{A}) = \mathcal{A} \otimes M_N$ and has a partial matrix unit system $\{f_{k,l}\}$ which is elementary in the sense that for each pair k, l the operator $(q_i \otimes I_N) f_{k,l} (q_j \otimes I_N)$ is nonzero for at most one pair i, j. The conjugacy class of each such subalgebra is determined by a subdigraph H of G and a projection q in $M_N(\mathcal{M})$. Because $\mathcal{M}$ is a II_1 factor all such possibilities arise. That is, given a projection q in $M_N(\mathcal{M})$ we can choose N large enough so that $\mathrm{trace}(q_i) \geq N^{-1}\mathrm{trace}(q)$, for each i. Then there is a natural partial matrix unit system $\{f_{k,l} : (k,l) \in E(G)\}$, in $M_N(\mathcal{A})$, with the elementary property above, such that $\mathrm{trace}(f_{k,k}) = \mathrm{trace}(q)$ for all k. If $\mathrm{trace}(q) = \alpha$ then denote the equivalence class of these digraph algebras (with $H = G$) by $[A_\alpha]$.

Let f be a partial isometry in $M_N(\mathcal{A})$, for some N, which normalises the subalgebra $M_N(\mathcal{A} \cap \mathcal{A}^*)$. Then f is elementary in the sense above. The principle involved here is that if a partial isometry of the form

$$\begin{bmatrix} 0 & 0 & v & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

normalises the block diagonal algebra of matrices

$$\begin{bmatrix} a & b & 0 & 0 & 0 & 0 \\ c & d & 0 & 0 & 0 & 0 \\ 0 & 0 & e & f & 0 & 0 \\ 0 & 0 & g & h & 0 & 0 \\ 0 & 0 & 0 & 0 & i & j \\ 0 & 0 & 0 & 0 & k & l \end{bmatrix}$$

then v or w is equal to zero. It follows that if $A \subseteq M_n(\mathcal{A})$ is an $\mathcal{A} \cap \mathcal{A}^*$-normalising digraph algebra for $\mathcal{A}$ then the partial matrix unit system for A is equivalent, by a unitary in $M_n(\mathcal{A} \cap \mathcal{A}^*)$, to a direct sum of subsystems for the algebras A_α identified above.

We now have the identification

$$H_n(\mathcal{A}; \mathcal{C}) = (\sum_{\alpha \in \mathbb{R}_+} \oplus H_n([A_\alpha]))/Q_n.$$

Identify each group $H_n([A_\alpha])$ with $H_n(\Delta(\mathcal{L}))$. As in the last proof we may replace Q_n by the subgroup corresponding to the relations of induced decompositions. This is the subgroup of $\sum_{\alpha \in \mathbb{R}_+} \oplus H_n([A_\alpha])$ generated by elements of the form

$$\sum_{\alpha} \oplus \delta_{\beta,\alpha} g \ - \sum_{\alpha} \oplus \delta_{\beta_1,\alpha} g - \sum_{\alpha} \oplus \delta_{\beta_2,\alpha} g$$

where $g \in H_n(\Delta(\mathcal{L}))$, $\beta = \beta_1 + \beta_2$ and where $\delta_{\beta,\alpha}$ is the Kronecker delta. It follows that $H_n(\mathcal{A};\mathcal{C}) = H_n(\Delta(\mathcal{L})) \otimes_{\mathbb{Z}} \mathbb{R}$ as desired.

(ii) The proof of (ii) is very similar to the proof of Theorem 2.1 and so we omit it. The regularity hypothesis for $\mathcal{C}$ is necessary because of the existence of singular masas, that is, masas with trivial normalisers. $\square$

Example 2.3 To see that the formula of Theorem 2.2 (i) is not valid when $\mathcal{M}$ is the I_∞ factor let $\mathcal{A} = A(D_4))$ be the subalgebra of $M_4(\mathbb{C})$ spanned by the matrix units $e_{13}, e_{14}, e_{23}, e_{24}$ and the standard diagonal subalgebra $\mathbb{C}^4$. This is the standard example of a matrix algebra with nontrivial Hochschild cohomology and the last theorem shows that $H_1(\mathcal{A};\mathcal{C}) = \mathbb{Z}$. Let $\mathcal{B}$ be the operator algebra on $\mathbb{C} \oplus (\mathbb{C}^4 \otimes \mathcal{H})$, where $\mathcal{H}$ is an infinite dimensional Hilbert space, consisting of operators of the form

$$\begin{pmatrix} \lambda & * \\ 0 & a \end{pmatrix}, \quad \text{with } \lambda \in \mathbb{C} \text{ and } \quad a \in \mathcal{A} \otimes \mathcal{L}(\mathcal{H}).$$

In the terminology of Gilfeather and Smith [7] this is the *cone algebra* of $\mathcal{A} \otimes \mathcal{L}(\mathcal{H})$. The algebra $\mathcal{B}$ is the reflexive operator algebra determined by a finite commutative projection lattice, with five atoms q_1, q_2, q_3, q_4, q_5 whose associated digraph for the order $\ll$, is a 4-cycle (for q_1, q_2, q_3, q_4) with an added vertex (for q_5) which receives four directed edges from each of the vertices of the 4-cycle. Here q_5 is the rank one projection onto the one dimensional summand, and q_i is $e_{i,i} \otimes I_{\mathcal{H}}$, for $1 \leq i \leq 4$. Although $H_1(\Delta(\mathcal{L}))$ is zero, clearly, the basic 1-cycle $(e_{1,1} \otimes I_{\mathcal{H}}, e_{2,2} \otimes I_{\mathcal{H}}, e_{3,3} \otimes I_{\mathcal{H}}, e_{4,4} \otimes I_{\mathcal{H}})$ gives a generator for $H_1(\mathcal{B}; \mathcal{B} \cap \mathcal{B}^*)$.

The next identification is in the context of limit algebras, one of our key motivating contexts for the formulation of stable homology.

Let $\mathcal{A} = \varinjlim(A(G_k), \phi_k)$ be the Banach algebra direct limit of a direct system of digraph algebras $A(G_k)$ with star-extendible injections $\phi_k : A(G_k) \to A(G_{k+1})$ which map standard matrix units to sums of standard matrix units. In particular, such maps are *regular* in the sense of the next section. For each $n \geq 0$ there is a natural induced group homomorphism

$$(\phi_k)_* : H_n(\Delta(G_k)) \to H_n(\Delta(G_{k+1}))$$

and an associated direct limit abelian group

$$\varinjlim(H_n(\Delta(G_k)), (\phi_k)_*).$$

Such limit groups have appeared in [2] and [20]. Let $\mathcal{D} = \varinjlim(\mathbb{C}^{|G_k|}, \phi_k)$ be the abelian C^*-subalgebra of $\mathcal{A}$, where $\mathbb{C}^{|G_k|}$ is the standard diagonal subalgebra of $A(G_k)$.

The following matricial variant of a fundamental fact for normalising partial isometries in AF C^*-algebras will be needed. The scalar case appears as Lemma 5.5 of [20].

Lemma 2.4 *Let $\mathcal{B}, \mathcal{D}$ be as above and let f be a partial isometry in $\mathcal{B} \otimes M_m$ which normalises $\mathcal{D} \otimes M_m$. Then $f = dw$ where d is a partial isometry in $\mathcal{D} \otimes M_m$ and w is a partial isometry in $\mathcal{B}_k$, for some k, which normalises the diagonal subalgebra $\mathcal{D}_k$.*

Proof: Let $\tilde{\mathcal{B}_k}$ be the algebra generated by $\mathcal{B}_k$ and $\mathcal{D}$ and let $P_n : \mathcal{B} \to \tilde{\mathcal{B}_k}$ be the natural projections, as given in Chapter 4 of [20] for example. In

particular P_n is the pointwise limit of maps $P_{n,r}$, $r = 1, 2, \ldots$, each of which has the form $P_{n,r}(b) = p_1 b p_1 + \ldots + p_r b p_r$ for some family of orthogonal projections in $\mathcal{D}$. This property shows that if $v \in \mathcal{B}$ is a partial isometry normalising $\mathcal{D}$ then so too is each operator $P_{n,r}(v)$, and hence so too is $P_n(v)$ itself. It follows that the map $P_n \otimes \mathrm{Id} : \mathcal{B} \otimes M_m \to \tilde{\mathcal{B}}_n \otimes M_m$ is defined in such a way that it follows that $(P_n \otimes \mathrm{Id})(f)$ is also a partial isometry which normalises $\mathcal{D} \otimes M_m$.

We can now argue exactly as in the proof in [20] for the scalar case $n = 1$.

Let q_n be the range projection of $(P_n \otimes \mathrm{Id})(f)$. Then q_n is a Cauchy sequence of projections in $\mathcal{D} \otimes M_m$ converging to ff^*. Since $\mathcal{D}$ is abelian it follows that there exists n_0 such that $q_n \in \tilde{D_{n_0}} \otimes M_m$ for all n. The lemma is straightforward in the special case $\mathcal{B} = \tilde{\mathcal{B}}_t$, and so it will be sufficient to prove that $(P_n \otimes \mathrm{Id})(f) \in \tilde{\mathcal{B}_{n_0}} \otimes M_m$ for all n, since from this it follows that $f \in \tilde{\mathcal{B}_{n_0}} \otimes M_m$.

Write $(P_n \otimes \mathrm{Id})(f) = (P_{n_0} \otimes \mathrm{Id})(f) + z$. Then $z = \sum c_i e_i$, a finite sum with coefficients c_i in $\mathcal{D} \otimes M_m$ and where each e_i is a standard matrix unit for $\mathcal{B}_n$ which is not subordinate to a standard matrix unit for $\mathcal{B}_{n_0}$. It follows that $(P_n \otimes \mathrm{Id})(z(P_{n_\circ} \otimes \mathrm{Id})(f)^*) = 0$ for $n \geq n_0$. Thus

$$ff^* = ((P_{n_\circ} \otimes \mathrm{Id})(f) + z)((P_{n_\circ} \otimes \mathrm{Id})(f) + z)^*$$

$$= q_{n_0} + zz^* + z((P_{n_\circ} \otimes \mathrm{Id})(f))^* + (P_{n_\circ} \otimes \mathrm{Id})(f)z^*$$

and so

$$(P_n \otimes \mathrm{Id})(ff^*) = (P_n \otimes \mathrm{Id})(ff^*) + (P_n \otimes \mathrm{Id})(zz^*).$$

Thus $(P_n \otimes \mathrm{Id})(zz^*) = 0$. Let $n \to \infty$ and we obtain $(P \otimes \mathrm{Id})(zz^*) = 0$. Since $(P \otimes \mathrm{Id})$ is a faithful expectation $z = 0$ as desired. □

Theorem 2.5 *Let $\mathcal{A}$ be the operator algebra $\varinjlim (A(G_k), \phi_k)$ with regular embeddings and diagonal subalgebra $\mathcal{D}$, as above. Then, for each $n \geq 0$, the*

stable homology group $H_n(\mathcal{A};\mathcal{D})$ *is isomorphic to the limit homology group* $\varinjlim(H_n(\Delta(G_k))),(\phi_k)_*)$.

Proof: Let $A \subseteq M_\infty(\mathcal{A})$ be a $\mathcal{D}$-normalising digraph algebra for $\mathcal{A}$ with a partial matrix unit system $\{f_{i,j} : (i,j) \in I_A\}$ which generates a full matrix unit system $\{f_{i,j} : 1 \leq i,j \leq m\}$ in $M_\infty(\mathcal{B})$, where $\mathcal{B}$ is the AF C^*-algebra generated by $\mathcal{A}$. Without loss of generality assume that the digraph of A is connected. From Lemma 2.4 it follows that the full system $\{f_{i,j}\}$ is unitarily equivalent, by a unitary in $M_\infty(\mathcal{D})$ to a system $\{g_{i,j}\}$ where, for some integer $k > 0$, each $g_{i,j}$ is a sum of the standard matrix units of the subalgebra $M_k(C^*(A(G_k)))$ of $M_\infty(\mathcal{B})$. Here we identify $A(G_k)$ and its generated C^*-algebra with its image in $\mathcal{A}$ and $C^*(\mathcal{A})$ respectively. It follows, as in the proof of Theorem 2.1, that

$$H_n(\mathcal{A};\mathcal{D}) = (\sum_k \oplus H_n([M_k(A(G_k))]))/Q_n$$

and so

$$H_n(\mathcal{A};\mathcal{D}) = (\sum_k \oplus H_n([A(G_k)]))/Q_n.$$

Furthermore in the second quotient expression we may assume that Q_n is the set of relations for the standard inclusions and induced decompositions amongst the set of digraph algebras $M_k(A(G_k))$.

Let η be the natural group homomorphism from the direct limit group $\mathcal{G}$ say, to $H_n(\mathcal{A};\mathcal{D})$. This is well-defined, because the relations Q_n include those relations coming from the given injections ϕ_k. On the other hand, suppose that $h \in H_n([A(G_k)])$ and $h \in Q_n$. Then there exists $k_1 > k$ so that g is a finite sum of terms of the form $g - \theta(g)$ and $g - \theta'(g) - \theta''(g)$ associated with the given inclusions $A(G_p) \to A(G_{k_1})$, for $1 \leq p \leq k$. Thus, viewed as a member of $A(G_k)$, g is the zero element. Hence η is injective and surjective. $\square$

The following examples can be obtained readily with the help of the theorems above.

Example 2.6 Let $\mathcal{A}$ be a strongly maximal triangular subalgebra of the AF C^*-algebra $\mathcal{B}$. (See [17], [20].) Then $H_1(A(D_4)\otimes\mathcal{A}) = K_0(\mathcal{A}\cap\mathcal{A}^*)$. Here the unique masa $\mathcal{A}\cap\mathcal{A}^*$ is understood and suppressed from the notation.

On the other hand let $A(D_4)\star\mathcal{A}$ be the lexicographic product (cf. [23]) given by

$$(A(D_4)\cap A(D_4)^*)\otimes\mathcal{A} \ + \ A(D_4)^0\otimes\mathcal{B},$$

where $A(D_4)^0$ is the kernel of the diagonal expectation onto the diagonal algebra $A(D_4)\cap A(D_4)^*$. This algebra is triangular, with a unique masa, and $H_1(A(D_4)\star\mathcal{A}) = K_0(\mathcal{B})$.

Example 2.7 Let

$$\phi_k : A(D_4)\otimes(M_{3^k}\oplus M_{3^k}) \to A(D_4)\otimes(M_{3^{k+1}}\oplus M_{3^{k+1}})$$

be the embedding $\phi\otimes id_{M_{3^{k-1}}}$, where ϕ is the embedding given before Definition 3.3. (Identify $(M_{3^k}\oplus M_{3^k})$ with $((M_3\oplus M_3)\otimes M_{3^{k-1}}$ etc.) Let $\mathcal{A}$ be the associated unital digraph limit algebra. Then, with respect to the natural diagonal subalgebra $\mathcal{C}$,

$$H_1(\mathcal{A};\mathcal{C}) \ = \ \varinjlim(\mathbb{Z}^2, \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}) \ = \ \mathbb{Z}^2.$$

There are a number of interesting connections between $H_n(\mathcal{A};\mathcal{C})$ and automorphism, derivations and Hochschild cohomology. The following theorem

is an example of this. Related assertions (with similar proofs) can be found in [2] [19] and [24].

Let $\mathcal{C}$ be a maximal abelian algebra in $\mathcal{A}$ and write $\mathrm{Aut}_{\mathcal{C}}(\mathcal{A})$ for the corresponding group of Schur automorphisms of $\mathcal{A}$. This is the group of automorphisms α for which $\alpha(c_1ac_2) = c_1\alpha(a)c_2$ for all c_1, c_2 in $\mathcal{C}$ and a in $\mathcal{A}$. If A is a $\mathcal{C}$-normalising digraph algebra for $\mathcal{A}$ then write $\tilde{A}$ for the algebra generated by A and $\mathcal{C}$. We say that a Schur automorphism is *locally* $\mathcal{C}$*-inner* if the restriction to each such subalgebra $\tilde{A}$ is inner.

Theorem 2.8 *Let $\mathcal{C}$ be a maximal abelian subalgebra of the operator algebra $\mathcal{A}$ and suppose that $H_1(\mathcal{A};\mathcal{C}) = 0$. Then every Schur automorphism in $Aut_{\mathcal{C}}(\mathcal{A})$ is locally $\mathcal{C}$-inner.*

A locally $\mathcal{C}$-inner automorphism need not be inner even for approximatelty finite C*-algebras and their regular subalgebras. (See, for example, Remark 2 of [22].) Nevertheless such automorphisms are often approximately inner in the sense of being approximable in the point-norm topology by inner (Schur) automorphisms. Thus, in rough parallel with the weakly closed theory developed in [24], it seems to be the case that there is a close connection between stable homology with respect to regular maximal abelian self-adjoint subalgebras, and the (norm) *essential* Hochschild cohomology arising when boundaries are replaced by their point norm closures - the essential boundaries.

3 Regular inclusions and $K_0 \oplus H_*$ uniqueness

The following distinguished class of embeddings is studied in [20], [19] and [21].

Definition 3.1 [21] A star-extendible algebra homomorphism between digraph algebras is said to be *regular* if it is (inner) unitarily equivalent to a direct sum of multiplicity one star-extendible embeddings.

A multiplicity one star-extendible embedding $A(G) \to A(H)$ is a restriction of a star homomorphism $C^*(A(G)) \to C^*(A(H))$ which is of multiplicity one. In particular every star homomorphism between self-adjoint digraph algebras is automatically regular. On the other hand there are, in general, a myriad of star-extendible homomorphisms between digraph algebras, and the regular embeddings form the most natural subclass. Between two digraph algebras there are only finitely many (inner) unitary equivalence classes of regular homomorphisms, and, for elementary algebras, these classes may be represented by diagrams at the level of digraphs. Bratteli diagrams form a degenerate case. The terminology "regular" is used because direct systems of regular embeddings provide limit algebras possessing a distinguished maximal abelian self-adjoint subalgebra which is regular in the usual sense that the normaliser of the masa generates the algebra.

An important aspect of regular morphisms is that they are the correct class of maps to consider with regard to the functoriality of stable homology; each regular homomorphism $\phi : A(G) \to A(H)$ induces a group homomorphism $\phi_* : H_n(A(G)) \to H_n(A(H))$. Here we have written $H_n(A(G))$ for $H_n(A(G); C)$ where C is any maximal abelian subalgebra of $A(G)$. This is a well-defined move since each such masa is unique up to inner unitary

equivalence.

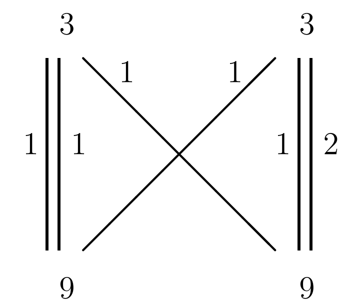
If we focus attention on a stable family of digraph algebras of the form $A(G) \otimes M_n, \quad n = 1, 2....,$ where G is a fixed digraph, then the following class of regular embeddings is particularly natural. As we shall see these rigid embeddings appear naturally in the construction of limit algebras with interesting homology. Furthermore, for various stable families we can classify associated rigid inclusions in terms of the induced map on $K_0 \oplus H_*$.

Definition 3.2. [20] (i) Let G be a connected digraph. A *rigid embedding* $A(G) \otimes M_n \to A(G) \otimes M_m$ is a regular embedding which is unitarily equivalent to a direct sum of embeddings $\theta \otimes \psi$ where $\psi : M_n \to M_m$ is a multiplicity one C^*-algebra algebra injection and $\theta : A(G) \to A(G)$ is an automorphism induced by a digraph automorphism.

(ii) A general rigid embedding $A(G) \otimes B_1 \to A(G) \otimes B_2$, with B_1, B_2 finite-dimensional C^* -algebras, is a star-extendible embedding for which the partial embeddings are rigid.

The unitary equivalence class of a rigid embedding can be indicated by a (unique) labelled Bratteli diagram in which each edge from a vertex i of level one to vertex j of level two indicates a multiplicity one partial rigid embedding, and the labelling of the edge indicates the particular automorphism θ used in the embedding.

For example, let θ_1 and θ_3 be the identity and rotation automorphisms of $A(D_4)$, and let θ_2 and θ_4 be the two reflections. The diagram



indicates the rigid embedding

$$\phi : A(D_4) \otimes (M_3 \ \oplus \ M_3) \to A(D_4) \otimes (M_9 \ \oplus \ M_9).$$

where all multiplicity one component embeddings have the identity automorphism excepting that for the edge labelled with a 2, which is the reflection θ_2. One can verify that (with natural identifications of the homology groups) ϕ induces maps $H_0\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ and $H_1\phi : \mathbb{Z}^2 \to \mathbb{Z}^2$ given by

$$H_0\phi = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{and} \quad H_1\phi = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}.$$

Definition 3.3 (i) A *cycle algebra*, or *2m-cycle digraph algebra*, is a digraph algebra of the form $A(D_{2m}) \otimes B$ where D_{2m} is the $2m$-cycle digraph and where B a finite-dimensional C^* -algebra.
(ii) If $A_1 \subseteq A_2$ are $2m$-cycle digraph algebras, then the inclusion is said to be *rigid* if the inclusion map is a rigid embedding.

The following proposition is elementary but it is a direct counterpart to the important fact that inclusions of finite-dimensional C^*-algebras are determined up to inner conjugacy by their induced K_0 maps.

Proposition 3.4 *A rigid embedding between cycle algebras is determined up to inner unitary equivalence by the induced maps between the K_0 groups and the first stable homology groups.*

Proof: Let D_{2m} be a $2m$-cycle digraph with receiving vertices labelled $v_1, v_3, \ldots, v_{2m-1}$ and emmitting vertices $v_2, v_4, \ldots, v_{2m}$. Let $\theta_1, \theta_3, \ldots, \theta_{2m-1}$ be the rotation automorphisms of D_{2m} such that $\theta_j(v_1) = v_j$, and let $\theta_2, \theta_4, \ldots, \theta_{2m}$ be the reflection automorphisms $\theta_{2j} = \eta \circ \theta_{2j-1}, 1 \leq j \leq m$, where η is the reflection fixing v_1. Write θ_k also for the automorphisms of $A(D_{2m})$ induced by these graph automorphisms.

A rigid embedding $\phi : A(D_{2m}) \otimes M_p \to A(D_{2m}) \otimes M_q$ is unitarily equivalent to the direct sum $r_1\theta_1 + \ldots + r_{2m}\theta_{2m}$ where we abuse notation and write $r_k\theta_k$ for the orthogonal direct sum of r_k copies of the embeddings $\theta_k \otimes id$. Clearly the $2m$-tuple $r_1, \ldots, r_{2m}$ is a complete invariant for the unitary equivalence class of ϕ. It will be enough to show that the inner equivalence class of ϕ is determined by this 2m-tuple.

The map $K_0\phi$, under the natural identification of the K_0 groups, has the form $X + JY$ where $X = X(r_1, r_3, \ldots, r_{2m-1})$ is the Laurent matrix

$$\begin{bmatrix} r_1 & 0 & r_{2m-1} & . & . & . & . & r_3 & 0 \\ 0 & r_1 & 0 & . & . & . & . & 0 & r_3 \\ r_3 & 0 & r_1 & . & . & . & . & r_5 & 0 \\ 0 & r_3 & 0 & . & . & . & . & 0 & r_5 \\ . & . & . & . & . & . & . & . & . \\ r_{2m-1} & 0 & . & . & . & . & . & r_1 & 0 \\ 0 & r_{2m-1} & . & . & . & . & . & 0 & r_1 \end{bmatrix},$$

where Y is the Laurent matrix $X(r_2, r_4, \ldots, r_{2m})$, and where J is the matrix

$$\begin{bmatrix} 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & . & . & 0 & 1 \\ 0 & 0 & . & . & 1 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & . \\ 0 & 1 & . & . & 0 & 0 \end{bmatrix}.$$

On the other hand the map $H_1\phi : \mathbb{Z} \to \mathbb{Z}$, under the natural identification of the H_1 groups, is $H_1\phi = [\delta]$ where

$$\delta = (r_1 + r_3 + \ldots + r_{2m-1}) - (r_2 + r_4 + \ldots + r_{2m}).$$

The proposition will be proven if we show that the two matrices $K_0\phi$ and $H_1\phi$ determine the coefficients $r_1, \ldots, r_{2m}$. To this end let $\pi : M_{2m} \to M_{2m}$ be the natural projection onto the Laurent matrices obtained by averaging the $2m$ entries of each of the m odd "diagonals" and replacing the other diagonals with zeros. Note that if X is a Laurent matrix then $\pi(JX)$ is a multiple of the "all ones" matrix $Z = X(1, 1, \ldots, 1)$. It follows that application of π to the matrix $X + JY$ determines the components X, Y up to a multiple of Z. That is, the ordered sets $\{r_1, r_3, \ldots, r_{2m-1}\}$, and $\{r_2, r_4, \ldots, r_{2m}\}$ are determined up to a common additive constant. But now the fact that the difference δ is given by $H_1\phi$ leads to the determination of $r_1, \ldots, r_{2m}$.
$\square$

Corollary 3.5 *Let A_1, A_2, and A be $2m$-cycle digraph algebras with $A_1 \subseteq A$, $A_2 \subseteq A$ where the inclusions are rigid. Then A_1 and A_2 are inner conjugate if and only if the inclusion maps induce the same maps between the K_0 groups and between the first stable homology groups.*

Definition 3.6 Let G be a digraph and let Θ be a subset of $\mathrm{Aut}(G)$. Then Θ is said to have the $K_0 \oplus H_1-$*uniqueness property* if the rigid embeddings from $A(G) \otimes M_p$ to $A(G) \otimes M_q$ which are associated with Θ are determined up to inner conjugacy by the induced maps on K_0 and H_1. The $K_0 \oplus H_*$ -*uniqueness property* is defined similarly.

As part of the general homology programme for limit algebras indicated in [20] it is of interest to determine contexts (G, Θ) which have the $K_0 \oplus H_*$ -uniqueness property. This gives a starting point for classifications of non-self-adjoint limit algebras in the style of Elliott's classification of AF C^*-algebras.

Example 3.7 Suspensions Let $K_n^i,\ i = 1, 2,$ be complete digraphs on n vertices. Define the *n-point suspension* of the digraph algebra $A = A(G)$ to be the digraph algebra S_nA with graph S_nG where the vertex and edge sets are given by

$$V(S_nG) = V(K_n^1) \cup V(K_n^2) \cup V(G),$$

$$E(S_nG) = E(K_n^1) \cup E(K_n^2) \cup E(G) \cup E$$

where $E = \{(v^i, w) : w \in V(G), v^i \in V(K_n^i), i = 1, 2\}$. Let G_1, G_2 be connected. A regular embedding $\phi : A(G_1) \to A(G_2)$ of multiplicity r induces a natural regular embedding $S_k\phi : S_k(A(G_1)) \to S_{kr}(A(G_2))$ which respects the north pole and south pole summands of the suspension algebras. This suspended embedding is uniquely determined up to inner conjugacy. From simplicial homology theory it follows that for each order t the suspended embedding $S_k\phi$ induces a homomorphism of the stable homology groups of

order $t+1$, and this homomorphism may be identified with the homomorphism of the homology groups of order t induced by ϕ. It follows that the homological classifications in this paper of various families of embeddings admit immediate higher order extensions to the classification of the associated pole preserving embeddings of the suspension algebras.

Example 3.8 Discrete Tori. The discrete tori algebras are the digraph algebras

$$A(D_{2m_1}) \otimes \ldots \otimes A(D_{2m_s})$$

whose underlying digraphs are the direct products of cycle digraphs. The full group of rigid automorphisms of these algebras fails to have the $K_0 \oplus H_*$-uniqueness property. To see this consider the rigid embeddings

$$\phi,\ \psi: \ A(D_4) \otimes A(D_4) \rightarrow \ A(D_4) \otimes A(D_4) \otimes M_{12}$$

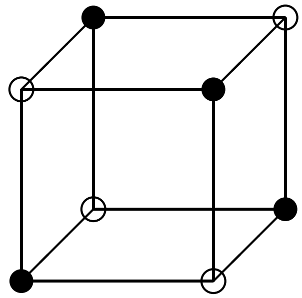
given by

$$\phi = ((2\theta_1 \oplus \theta_3) \otimes (\theta_1 \oplus \theta_3)) \oplus ((\theta_1 \oplus 2\theta_3) \otimes (\theta_2 \oplus \theta_4)),$$

$$\psi = ((\theta_1 \oplus 2\theta_3) \otimes (\theta_1 \oplus \theta_3)) \oplus ((2\theta_1 \oplus \theta_3) \otimes (\theta_2 \oplus \theta_4)).$$

Then $K_0\phi$ and $K_0\psi$ coincide with $3X \otimes X$, where X is the "all ones" matrix $X(1,1,1,1)$. Also one can verify that $H_0\phi = H_0\psi = [12]$, $H_2\phi = H_2\psi = [0]$, and $H_1\phi = H_1\psi = 0$, the zero map from $\mathbb{Z}^2$ to $\mathbb{Z}^2$. Thus $(K_0 \oplus H_*)\phi = (K_0 \oplus H_*)\psi$ and yet the injections are not inner conjugate.

Example 3.9 The Cube Algebra. Define the *cube algebra* to be the digraph algebra in M_8 which is associated with the following digraph, which we denote as Cu.



This may be regarded as a three dimensional variant of the 4-cycle graph which appears on each face of the cube. The full automorphism group Aut (Cu) has 24 elements corresponding to the 24 permutations of the receiving vertices. Note that there is a unique directed graph automorphism of Cu for each such permutation. Thus Aut (Cu) has order 24, and a general rigid embedding $\phi : A(Cu) \otimes M_n \to A(Cu) \otimes M_m$ has an inner unitary conjugacy class which is determined by the ordered set $\{r_1, \ldots, r_{24}\}$ corresponding to the multiplicities of the types of partial rigid embeddings. Furthermore, it follows that in the direct sum decomposition

$$K_0\phi = K_0^r\phi \oplus K_0^e\phi,$$

corresponding to the receiving and emmitting summands, the linear system in the unknowns $\{r_1, \ldots, r_{24}\}$ arising from the equation $K_0^r\phi = K_0^r\psi$, with ψ given, has the same rank as the system for the full equation $K_0\phi = K_0\psi$. Thus, knowledge of the 4 by 4 matrix $K_0\phi$ leads to 16 equations for $\{r_1, \ldots, r_{24}\}$. We have $H_1(A(Cu) \otimes M_n) = \mathbb{Z}^5$, and so 25 more equations are provided by $H_1\phi$ giving a system of 41 linear equations in 24 unknowns. Curiously, (computer assisted) calculation shows that the coefficient matrix of this system has rank 23 and so the full automorphism group for the cube algebra just misses having the $K_0 \oplus H_*$ - uniqueness property. This can be seen directly by considering the multiplicity 12 embedding which is a direct sum of the rotations and the multiplicity 12 embedding which is the direct

sum of the rest. Both induce the zero map on H_1 and both have the same K_0 map.

On the other hand, proper subgroups of Aut (Cu) do have this uniqueness property. In particular, this is the case for the group of 12 orientation preserving symmetries of the cube digraph. Calculation shows that the coefficient matrix in this simpler case is the following.

Coefficient Matrix arising from Rotations of Cu

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
$$

The rank of the matrix is 12. The submatrix arising from the K_0 data alone is the 16 by 12 submatrix formed by the first 16 rows, and this has rank 10. Thus, as in the case of the cycle algebras, the stable homology information is really needed. We have

Theorem 3.10 *Let $\mathcal{A}$ be the cube algebra $A(Cu) \otimes M_n$. Let $\mathcal{F}$ be the family of subalgebras of $\mathcal{A}$ which are completely isometrically isomorphic to a cube algebra $A(Cu) \otimes M_r$, for some r, and for which the inclusion map is a rigid embedding associated with rotations. Then the algebras in $\mathcal{F}$ are classified up to inner conjugacy by the following two invariants.*

(i) the inclusion induced map between the scaled K_0 groups,

(ii) the induced map between the first stable homology groups.

As we can see, even for simple digraph algebras the $K_0 \oplus H_*$ data can generate a large system for the unknown multiplicities of the components. It is of interest therefore to discover general combinatorial principles that can assist with rank determination.

4 Regular Inclusions in AF algebras

We now consider regular inclusions in the context of C^*-algebras.

The following terminology will be useful. Let $\mathcal{A} = \varinjlim(A(G_k), \phi_k)$ be a limit algebra as in Theorem 2.4 with diagonal subalgebra $\mathcal{D}$. Refer to such an algebra as a *regular digraph limit algebra* and say that $\mathcal{D}$ is a *regular canonical masa*, both of $\mathcal{A}$ and the superalgebra $\mathcal{B} = C^*(\mathcal{A})$. In the self-adjoint context, $\mathcal{B} = \mathcal{A}$, for which we may asssume that each G_k is a union of complete digraphs, it is known that a regular canonical masa is independent of the presentation of $\mathcal{A}$, in the following sense: if $\mathcal{D}$ and $\mathcal{D}'$ are two such masas in $\mathcal{B}$, arising from different presentations of $\mathcal{B}$, then there is an approximately inner automorphism $\alpha : \mathcal{B} \to \mathcal{B}$ such that $\alpha(\mathcal{D}) = \mathcal{D}'$. This uniqueness theorem is due to Kreiger (see Renault [25]) and a direct proof is given in [20]. It would be very interesting to know if regular canonical masas were unique in this way in general (cf. Remark 1.8). The following non-self-adjoint generalisation is straightforward.

Theorem 4.1 *Let $\mathcal{A} = A(G) \otimes B$ where B is an AF C^*-algebra and $A(G)$ is a digraph algebra. If $\mathcal{C}$ and $\mathcal{C}'$ are regular canonical masas of $\mathcal{A}$ then there exists an approximately inner automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ with $\alpha(\mathcal{C}) = \mathcal{C}'$.*

Proof: We give a proof for the case when B is an UHF C^*-algebra - the setting for Theorem 4.5 - and leave the reader to make the minor changes necessary for the general case.

Assume that G is connected. Let $\{h_{i,j}\}$ be a partial matrix unit system for $A(G)$. Note first that a masa in $A(G) \otimes B$ is inner unitarily equivalent to one of the form $h_{1,1} \otimes C^{(1)} + \ldots + h_{r,r} \otimes C^{(r)}$ where $r = |G|$ and where each $C^{(k)}$ is a regular canonical masa in B. We show now that we can further

arrange that the masas $C^{(k)}$ coincide and are equal to a regular canonical masa, C say, in the C^*-algebra B. Since $\mathcal{C}'$ is similarly conjugate to a masa of the form $h_{1,1} \otimes C' + \ldots + h_{r,r} \otimes C'$ for some regular canonical masa C' in B, the theorem follows readily from the self-adjoint case.

The masa $\mathcal{C}$ can be described in the following way. There is a matrix unit system $\{e_{i,j}^{(k)}\}$ for $\mathcal{B} = C^*(\mathcal{A})$ such that for each k the finite system $\{e_{i,j}^{(k)}\}$ is a full matrix unit system for a unital matrix subalgebra $\mathcal{B}_k$ of $\mathcal{B}$, and the following properties hold :

(i) for fixed k each partial isometry $e_{p,q}^k$ is a sum of some of the matrix units of $\{e_{i,j}^{(k+1)}\}$,

(ii) the matrix algebra inclusions $\mathcal{B}_k \subseteq \mathcal{B}_{k+1}$ are unital,

(iii) $\mathcal{C}$ is the closed span of a chain of masas $\mathcal{C}_k \subseteq \mathcal{B}_k$ where $\mathcal{C}_k = \mathrm{span}\{e_{i,i}^{(k)}\}$,

(iv) $\mathcal{A} \cap \mathcal{B}_k$ is spanned by some of the matrix units of $\{e_{i,j}^{(k)}\}$, including all the diagonal matrix units $\{e_{i,i}^{(k)}\}$.

Without loss of generality assume that $h_{j,j} \otimes 1$ lies in $\mathcal{C}_1$ for each j. Then each $h_{j,j} \otimes 1$ is the sum of the same number of minimal diagonal matrix units in the set $\{e_{i,i}^{(1)}\}$. It follows that there is a partial isometry v in $\mathcal{B}_1$ which is a sum of matrix units in the set $\{e_{i,j}^{(1)}\}$ and has initial projection $h_{2,2} \otimes 1$ and final projection $h_{1,1} \otimes 1$. Necessarily $v = h_{1,2} \otimes w$ for some partial isometry w in B. Since it is a sum of matrix units it must normalise the masa $\mathcal{C}$ and so $v(h_{2,2} \otimes C^{(2)})v^* = h_{1,1} \otimes C^{(1)}$ and hence $h_{1,1} \otimes wC^{(2)}w^* = h_{1,1} \otimes C^{(1)}$. Using such elements w construct a unitary operator in the diagonal algebra $\sum h_{i,i} \otimes B$ which conjugates $\mathcal{C}$ to a masa of the desired form. $\square$

As in the finite-dimensional setting, the following definition is now well-defined and natural.

Definition 4.2 For $\mathcal{A} = A(G) \otimes B$ as above define the *stable regular (partial isometry) homology* of $\mathcal{A}$ to be the groups $H_n(\mathcal{A}) = H_n(\mathcal{A};\mathcal{C})$, for $n = 0, 1, 2, ...$, where $\mathcal{C}$ is a regular canonical masa of $\mathcal{A}$.

Definition 4.3 Let $\mathcal{A}$, $\mathcal{A}'$ be regular digraph limit algebras. Then

(i) an algebra homomorphism $\mathcal{A} \to \mathcal{A}'$ is said to be *regular* if there exist regular canonical masas $\mathcal{C} \subseteq \mathcal{A}$, $\mathcal{C}' \subseteq \mathcal{A}'$ such that $\beta(\mathcal{C}) \subseteq \mathcal{C}'$ and $\beta(N_{\mathcal{C}}(\mathcal{A})) \subseteq N_{\mathcal{C}'}(\mathcal{A}')$ where $N_{\mathcal{C}}(\mathcal{A})$ is the partial isometry normaliser of $\mathcal{C}$ in $\mathcal{A}$.

(ii) If $\mathcal{A}'' \subseteq \mathcal{A}$ then $\mathcal{A}''$ is said to be a *regular subalgebra* if it is star-extendibly isomorphic to a regular digraph limit algebra and the inclusion map is regular.

The simplest regular subalgebras are the closed subalgebras $\mathcal{A}''$ such that $\mathcal{C} \subseteq \mathcal{A}'' \subseteq \mathcal{A}$ for some regular canonical masa $\mathcal{C}$ of $\mathcal{A}$. These may be thought of as the multiplicity one subalgebras. They are automatically regular digraph limit algebras, and they are described in terms of subrelations of the approximately finite semigroupoid $R(\mathcal{A};\mathcal{C})$ associated with $\mathcal{C}$. For details see Chapter 7 of [20]. On the other hand the unital inclusion $A(G) \otimes B \to A(G) \otimes B \otimes M_n$ given by $a \to a \otimes 1_n$ is a regular inclusion of finite multiplicity n in the sense that the commutant of the range is isomorphic to M_n .

In general, in addition to the index of the inclusion, we need K-theoretic data, stable homology data, and perhaps other invariants in order to determine the conjugacy class.

Extending the earlier usage, say that an embedding $\alpha : A(G) \otimes B \to A(G) \otimes B'$ is *rigid* if there is an identification $B' = M_n \otimes B$ such that

$\alpha(a) = \phi(a) \otimes id_B$ where ϕ is a rigid embedding. The multiplicity of α is defined to be the multiplicity of ϕ .

In fact such embeddings and their multiplicities may be characterised intrinsically, without reference to a postulated tensor decomposition, in terms of the fundamental topological binary relation $R(\mathcal{A}')$ for the pair $(\mathcal{A}', \mathcal{C}')$. This fact is not needed below but we nevertheless indicate this characterisation in the case of the 4-cycle $G = D_4$.

Let v_1, v_2, v_3, v_4 be the images of $e_{1,3}\otimes 1, e_{1,4}\otimes 1, e_{2,3}\otimes 1, e_{2,4}\otimes 1$ under the rigid embedding α. For each point x in the Gelfand space $M(\mathcal{C}')$, which is dominated by the initial projection of one of these images, the partial isometries v_i determine a subgraph of $R(\mathcal{A}')$. A simple compactness argument shows that the embedding is rigid if and only if each such subgraph is a nondegenerate copy of G in the sense of being equivalent to the canonical copies of G. (Of course, while all these copies of G are equivalent this equivalence need not respect the labellings inherited from the partial isometries v_i.)

We now generalise Proposition 3.4 and classify the rigid embeddings betweeen cycle algebras of the form $\mathcal{A} = A(D_{2m}) \otimes B$ where B is a UHF C^*-algebra. The following extra homological invariant is needed.

Definition 4.4 The *scale* of the stable homology group $H_1(\mathcal{A}; \mathcal{C})$ is the subset $\Sigma_1(\mathcal{A}; \mathcal{C})$ of elements arising from cycles associated with partial matrix unit systems $\{e_{i,j}\}$ with $e_{i,i} \in \mathcal{A}$ for all i .

In the case of the cycle algebras $\mathcal{A} = A(D_{2n}) \otimes B$ we may write $\Sigma_1(\mathcal{A})$ for the scale and there is a natural identification

$$(H_1(\mathcal{A}), \Sigma_1(\mathcal{A})) = (K_0(B), [-1_B, 1_B])$$

where $(K_0(B), 1_B) = (\mathbb{Q}(n), 1)$ and $\mathbb{Q}(n)$ is the subgroup of $\mathbb{Q}$ associated with the generalised integer n for B. Define the *scale* of $K_0\mathcal{A} \oplus H_1\mathcal{A}$ to be the subset of $\Sigma(\mathcal{A}) \times \Sigma_1(\mathcal{A})$ consisting of the pairs $([p], \sigma)$ where $\sigma \in \Sigma_1$ arises from a cycle associated with a partial matrix unit system $\{e_{i,j}\}$ with $[p] = [e_{i,i}]$.

Theorem 4.5 *Let $\mathcal{A}_1 = A(D_{2n}) \otimes B$ and $\mathcal{A}_2 = A(D_{2n}) \otimes B'$ where B and B' are UHF C^*-algebras, and let $\alpha_i : \mathcal{A}_1 \to \mathcal{A}_2, i = 1, 2,$ be rigid embeddings. Then α_1 and α_2 are inner unitarily equivalent if and only if the following conditions hold.*

(i) α_1 and α_2 have the same multiplicity.

(ii) α_1 and α_2 induce the same scaled group homomorphisms from $K_0\mathcal{A}_1 \oplus H_1\mathcal{A}_1$ to $K_0\mathcal{A}_2 \oplus H_1\mathcal{A}_2$.

Proof: The necessity of the conditions is straightforward.

For the converse we may assume, by replacing α_1 and α_2 by conjugate maps, that $\alpha_i = \phi_i \otimes id_B$ where $B' = M_m \otimes B$ for some integer m, which is greater than the multiplicities of α_1 and α_2, and where each map $\phi_i : A(D_4) \to A(D_4) \otimes M_m$ is a rigid embedding. Thus, in view of Proposition 3.4 it remains to show that the information of (i) and (ii) is sufficient to determine $K_0\phi_i$ and $H_1\phi_i$.

Let s be the generalised integer for B. Then $(K_0\mathcal{A}, \Sigma(\mathcal{A}))$ is identifiable with the $2n$-fold product

$$(\mathbb{Q}(s) \oplus \ldots \oplus \mathbb{Q}(s), [0,1]^{2n})$$

and

$$(H_1\mathcal{A}, \Sigma_1(\mathcal{A})) = (\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q}(s), [-1,1]) = (\mathbb{Q}(s), [-1,1]).$$

There are similar identifications for $(K_0\mathcal{A}', \Sigma(\mathcal{A}'))$ with ms in place of s and under these identifications it follows that $K_0\alpha_i$, as a $2n$ by $2n$ matrix, is equal to $q_iK_0\phi_i$, where q_i is equal to the inverse of the multiplicity of ϕ_i . Furthermore, as a 1 by 1 matrix, $H_1\alpha_i$ is equal to $q_iH_1\phi_i$. By the hypotheses it follows that ϕ_1 and ϕ_2 have coincident $K_0 \oplus H_1$ data, as desired. □

Remark 4.6 As we have already mentioned it would be desirable to generalise Theorem 4.1 to general regular limits of digraph algebras. The essential obstacle for this is already present in the case of algebraic direct limits. Suppose that $\mathcal{A}$ is such a limit algebra with two digraph subalgebra chains

$$A_k \subseteq A_{k+1} \text{ and } A'_k \subseteq A'_{k+1}$$

for all $k = 1, 2, \ldots$, where $A_1, A_2, \ldots$ and $A'_1, A'_2, \ldots$ are digraph algebras, with dense union, for which the given inclusions are regular. In particular it is possible to choose partial matrix unit systems, in the usual sense, for the chains $\{A_k\}$ and $\{A'_k\}$, which in turn determine regular canonical masas, $\mathcal{C}$ and $\mathcal{C}'$ say, spanned by the diagonal matrix units. Choosing subsystems and relabelling we may assume furthermore that $A_k \subseteq A'_k \subseteq A_{k+1}$ for all k. If these inclusions are regular then it can be shown that $\mathcal{C}$ and $\mathcal{C}'$ are conjugate by an approximately inner automorphism of $\mathcal{A}$. (In particular it follows that the conjugacy class of $\mathcal{C}$ is determined by the chain $\{A_k\}$ and is independent of the choice of matrix unit system.) However examples can be constructed wherein these inclusions are not regular.

5 Limit algebras

The following discussion illustrates the use of $K_0 \oplus H_1$-uniqueness in the identification of limit algebras.

Consider the system $\mathcal{A}_1 \to \mathcal{A}_2 \to \dots$ consisting of 4-cycle digraph algebras

$$A(D_4) \oplus A(D_4) \to (A(D_4) \oplus A(D_4)) \otimes M_{20} \to (A(D_4) \oplus A(D_4)) \otimes M_{20^2} \to \dots \mathcal{A}.$$

Assume furthermore that this is a stationary direct system in which each embedding is a fixed rigid embedding similar to the type mentioned before Definition 3.3. That is, the k^{th} embedding of the system has the form $\phi_k = \phi \otimes \mathrm{id}_{k-1} : \mathcal{A}_1 \otimes M_{10^{k-1}} \to (\mathcal{A}_1 \otimes M_{10}) \otimes M_{10^{k-1}}$ where

$$\phi = \begin{bmatrix} \psi_1 & \psi_2 \\ \psi_3 & \psi_4 \end{bmatrix}$$

and where each partial embedding ψ_i is a rigid embedding of the form $r_1\theta_1 + \dots + r_4\theta_4$. (The coefficients r_k depend on i.) Make the additional restriction that

$$K_0\phi = \begin{bmatrix} 5 & 5 & 0 & 0 & 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 \\ 5 & 5 & 0 & 0 & 5 & 5 & 0 & 0 \\ 5 & 5 & 0 & 0 & 5 & 5 & 0 & 0 \\ 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 \\ 0 & 0 & 5 & 5 & 0 & 0 & 5 & 5 \end{bmatrix},$$

so that

$$H_0\phi = \begin{bmatrix} 10 & 10 \\ 10 & 10 \end{bmatrix},$$

and for convenience denote these matrices by T and S respectively. With these assumptions the stationary limit algebra $\mathcal{A}$ is determined by the 2×2 integral matrix $X = H_1\phi$. Write $\mathcal{A}_X$ for the algebra. For each of the partial embeddings ψ of ϕ there are six possibilities. In the notation of Proposition 3.4 these are

$$5\theta_1 + 5\theta_3,\ 4\theta_1 + 4\theta_3 + \theta_2 + \theta_4,\ 3\theta_1 + 3\theta_3 + 2\theta_2 + 2\theta_4,$$

$$2\theta_1 + 2\theta_3 + 3\theta_2 + 3\theta_4,\ \theta_1 + \theta_3 + 4\theta_2 + 4\theta_4,\ 5\theta_2 + 5\theta_4.$$

The induced homomorphisms on H_1 are the maps $\mathbb{Z} \to \mathbb{Z}$ with entries

$$10, 6, 2, -2, -6, -10,$$

respectively. These numbers form the so called homology range (in the terminology of [20] of a rigid embedding for $K_0\psi$ (and, by terminological extension, for ψ itself). There are thus 6^4 possibilities for the matrix X, and, a priori, a great many possibilities for the limit algebras $\mathcal{A}_X$. Note that all of these algebras induces the same inclusion

$$\mathcal{A}_X \cap \mathcal{A}_X^* \to C^*(\mathcal{A}_X).$$

Let us focus on two of these algebras, namely

$$\mathcal{A}_{\left[\begin{smallmatrix} 10 & 6 \\ 6 & 10 \end{smallmatrix}\right]} \text{ and } \mathcal{A}_{\left[\begin{smallmatrix} 6 & 2 \\ 2 & 6 \end{smallmatrix}\right]}.$$

This pair is of interest because, with respect to the natural masas,

$$H_1(\mathcal{A}_{\left[\begin{smallmatrix} 10 & 6 \\ 6 & 10 \end{smallmatrix}\right]}) = H_1(\mathcal{A}_{\left[\begin{smallmatrix} 6 & 2 \\ 2 & 6 \end{smallmatrix}\right]}) = \mathbb{Q}(2^\infty) \oplus \mathbb{Q}(2^\infty).$$

Coincidence of this homology suggests that the two limit algebras may be isomorphic, and indeed they are.

The method of proof in this rather typical stationary example is to make use of the $K_0 \oplus H_1-$uniqueness property to construct a commuting diagram linking the two systems for the algebras.

Proposition 5.1 *The 4-cycle limit algebras* $\mathcal{A}_{\left[\begin{smallmatrix} 10 & 6 \\ 6 & 10 \end{smallmatrix}\right]}$ *and* $\mathcal{A}_{\left[\begin{smallmatrix} 6 & 2 \\ 2 & 6 \end{smallmatrix}\right]}$ *are star-extendibly isomorphic.*

Proof: Let $\mathcal{A} = \varinjlim(\mathcal{A}_k, \phi_k)$, $\mathcal{A}' = \varinjlim(\mathcal{A}'_k, \phi'_k)$ be the respective systems for the algebras, as above, and let X and Y be their respective 2 by 2 integral matrices. Consider the commuting diagram

$$\begin{tikzcd}
H_1(\mathcal{A}_1) \arrow[r, "X"] \arrow[d, "\mathrm{id}"'] & H_1(\mathcal{A}_2) \arrow[rrd, "V_1"] & & \\
H_1(\mathcal{A}'_1) \arrow[r, "Y"'] \arrow[ru, "U_1"] & H_1(\mathcal{A}'_2) \arrow[r, "Y"'] & \cdots\cdots \arrow[r, "Y"'] & H_1(\mathcal{A}'_{2+j})
\end{tikzcd}$$

where $U_1 = X$. We wish to choose j large enough so that the matrix $V_1 = Y^{1+j}U_1^{-1}$ is an integral matrix belonging to the homology range of the map

$$\begin{tikzcd} K_0\mathcal{A}_2 \arrow[r, "T^j"] & K_0\mathcal{A}'_{2+j} \end{tikzcd}$$

Note that the homology range can be easily calculated from the matrix S^j. In fact $j = 2$ is the first index for which this occurs, with

$$V_1 = \begin{bmatrix} 24 & 8 \\ 8 & 24 \end{bmatrix} \qquad S^2 = \begin{bmatrix} 200 & 200 \\ 200 & 200 \end{bmatrix}$$

We can now simultaneously lift U_1 and T to a rigid embedding $\beta : \mathcal{A}'_1 \to \mathcal{A}_2$ and we can lift V_1 and T^2 to a rigid embedding $\alpha_1 : \mathcal{A}_2 \to \mathcal{A}'_4$. Furthermore since

$$K_0 \oplus H_1(\alpha_1 \circ \beta_1) = K_0 \oplus H_1(\phi'_3 \circ \phi'_2 \circ \phi'_1)$$

we may apply Proposition 3.4 and replace α_1 by an inner conjugate map so that

$$\alpha_1 \circ \beta_1 = \phi'_3 \circ \phi'_2 \circ \phi'_1$$

Consider next the diagram

$$\begin{tikzcd} H_1\mathcal{A}_2 \arrow[r, "X"] \arrow[d, "V_1"'] & \cdots\cdots \arrow[r, "X"] & H_1\mathcal{A}_{2+k} \\ H_1\mathcal{A}'_4 \arrow[urr, "U_2"'] & & \end{tikzcd}$$

We wish to choose k large enough so that the matrix $U_2 = X^k V_1^{-1}$ is an integral matrix lying in the homology range of T^k. It is clear that such a k exists for the following two reasons.

(i) the entries of T^k will eventually exceed in modulus the corresponding entries of $X^k V_1^{-1}$.

(ii) all entries of T^k and $X^k V_1^{-1}$ are congruent to zero mod 4 for sufficiently large k.

In fact the first value for which (i) and (ii) hold is $k = 4$ giving

$$X^4 V_1^{-1} = \begin{bmatrix} 1032 & 1016 \\ 1016 & 1032 \end{bmatrix}, \qquad S^4 = \begin{bmatrix} 80000 & 80000 \\ 80000 & 80000 \end{bmatrix}.$$

As before we can lift U_2 to a rigid homomorphism β_2 in such a way that we obtain a commuting triangle so that $\beta_2 \circ \alpha_1 = \phi_5 \circ \phi_4 \circ \phi_3 \circ \phi_2$. It is clear that the requirements of (i) and (ii) can always be met at further stages in the construction of the commuting diagram. In this way we obtain the desired commuting diagram

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[r] \arrow[d, "\alpha_1"'] & \mathcal{A}_{m_1} \arrow[r] \arrow[d, "\alpha_2"'] & \mathcal{A} \arrow[d, "\cdots \quad \alpha"'] \\
\mathcal{A}'_{n_1} \arrow[r] \arrow[ur] & \mathcal{A}'_{n_2} \arrow[r] \arrow[ur] & \mathcal{A}'
\end{tikzcd}$$

$\square$

The reader may notice that the stationary case above presents no difficulties with regard to the harmonisation of the homology coupling invariants given in Chapter 11 of [20]. Addressing this issue is just one of the tasks

necessary for a complete classification of rigid embedding limits of digraph algebras.

Using the method of the last proof one can obtain the following more general theorem.

Theorem 5.2 *Let $\mathcal{A}_X$ and $\mathcal{A}_Y$ be limit algebras, as above, associated with a pair of 2 by 2 matrices whose entries lie in the set* $\{10, 6, 2, -2, -6, -10\}$. *If the (diagonal masa) homology groups $H_1(\mathcal{A}_X)$ and $H_1(\mathcal{A}_Y)$ are isomorphic then $\mathcal{A}_X$ and $\mathcal{A}_Y$ are star-extendibly isomorphic operator algebras. Furthermore, for the algebras $\mathcal{A}_X$ with $X = \left[\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right]$ there are at most five isomorphism classes corresponding to the groups* $\mathbb{Q}(2^\infty)$, $\mathbb{Q}(6^\infty)$, $\mathbb{Q}(10^\infty)$, $\mathbb{Q}(2^\infty) \oplus \mathbb{Q}(2^\infty)$, $\mathbb{Q}(2^\infty) \oplus \mathbb{Q}(6^\infty)$.

References

[1] E. Christensen, *Derivations of nest algebras*, Math. Ann. 229 (1977), 155-161.

[2] K.R. Davidson and S.C. Power, *Isometric automorphisms and homology for non-self-adjoint operator algebras*, Quart.Math.J., 42 (1991), 271-292.

[3] M. Gerstenhaber and S.P. Schack, *Simplicial homology is Hochschild cohomology* J. Pure and Appl. Algebra, 30 (1983), 143-156.

[4] F. Gilfeather, *Derivations of certain CSL algebras,* J. Operator Th., 11 (1984), 91-108.

[5] F. Gilfeather, A. Hopenwasser, and D. Larson, *Reflexive algebras with finite width lattices: Tensor products, cohomology, compact perturbations,* J. Functional Anal., 55 (1984), 176-199.

[6] F. Gilfeather and R.L. Moore, *Isomorphisms of certain CSL algebras,* J. Functional Anal., 67 (1986) 264–291.

[7] F. Gilfeather and R.R. Smith, *Cohomology for operator algebras : Cones and Suspensions,* Proc. London Math. Soc., to appear.

[8] F. Gilfeather and R.R. Smith, *Cohomology for operator algebras : Joins,* preprint 1992.

[9] F. Gilfeather and R.R. Smith, *Operator algebras with arbitrary Hochschild cohomology*, Contemporary Math., 120 (1991), 33-40.

[10] A. Ya. Helemskii, The homology of Banach and topological algebras, Mathematics and its Applications (Soviet Series) vol 41, Kluwer, Dordrecht, Boston, London, 1986.

[11] B. E. Johnson, *Cohomology in Banach Algebras*, Memoirs A.M.S. 127 (1972).

[12] B.E. Johnson, R.V.Kadison and J.R. Ringrose, Cohomology of operator algebras III. Reduction to normal cohomology, Bull. Soc. Math. France 100 (1972), 73-96.

[13] R.V. Kadison and J.R. Ringrose, *Cohomology of operator algebras I, Type I von Neumnann algebras*, Acta Math. 126 (1971), 227-243.

[14] J. Kraus and S.D. Schack, *The cohomology and deformations of CSL algebras,* unpublished notes, 1988.

[15] E.C. Lance, *Cohomology and perturbations of nest algebras* Proc. London Math Soc., 43 (1981), 334-356.

[16] P.S. Muhly and B. Solel, *Subalgebras of groupoid C*-algebras*, J. für die Reine und Ange. Math. 402 (1989), 41–75.

[17] J. Peters, Y. Poon and B. Wagner, *Triangular AF algebras*, J. Operator Th., 23 (1990), 81–114.

[18] S.C.Power, *Classifications of tensor products of triangular operator algebras,* Proc. London Math. Soc., 61 (1990),571-614.

[19] S.C. Power, *Non-self-adjoint operator algebras and inverse systems of simplicial complexes,* J. fur der Reine und Angew. Math., 421 (1991), 43-61.

[20] S.C. Power, Limit algebras: an introduction to subalgebras of C*-algebras, Pitman Research Notes in Mathematics vol 278, Longman, 1992.

[21] S.C. Power, *Algebraic orders on K_0 and approximately finite operator algebras,* J. Operator Th., to appear .

[22] S.C. Power, *On the outer automorphism groups of triangular alternation limit algebras* J. of Functional Anal. 113 (1993), 462-471.

[23] S.C. Power, *Infinite lexicographic products of triangular algebras*, Bull. London Math. Soc., to appear.

[24] S.C. Power, *Homology for operator algebras I : Spectral homology for reflexive algebras*, preprint 1993.

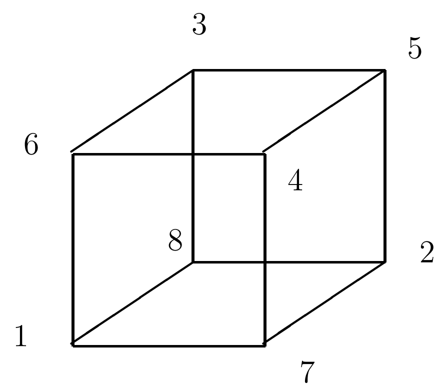
[25] J. Renault, A groupoid approach to C*-algebras, Lecture Notes in Math. No. 793, Springer Verlag, Berlin-Heidelberg-New York 1980.

[26] J. Taylor, *Homology and cohomology for topological algebras,* Advances in Mathematics, 9 (1972), 137-182.

Appendix 1

The Coefficient Matrix for the rotation embeddings of Cu

Label Cu in the following manner, where the receiving vertices are labelled 1,2,3,4.



The K_0 maps of the 12 multiplicity one embeddings associated with the 12 rotations of Cu are given by

$$T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad T_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$T_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_6 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$T_7 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad T_8 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad T_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$T_{10} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Consider the basis of $H_1(A(Cu)) = \mathbb{Z}^5$ given by the cycles

$$< 3,5 > + < 5,4 > + < 4,6 > + < 6,3 >,$$

$$< 6,2 > + < 2,8 > + < 8,1 > + < 1,6 >,$$

$$< 3,5 > + < 5,2 > + < 2,8 > + < 8,3 >,$$

$$< 5,4 > + < 4,7 > + < 7,2 > + < 2,5 >,$$

$$< 4,6 > + < 6,1 > + < 1,7 > + < 7,4 > .$$

Then the following matrices represent the corresponding H_1 maps of the 12 multiplicity one rotation embeddings.

$$S_1 = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix}, \quad S_3 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad S_5 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}, \quad S_6 = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \end{bmatrix},$$

$$S_7 = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad S_8 = \begin{bmatrix} 0 & 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \quad S_9 = \begin{bmatrix} 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

$$S_{10} = \begin{bmatrix} 0 & -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}, \quad S_{11} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_{12} = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix}.$$

The coefficient matrix in section 3 arises from the 41 equations in the multiplicities $r_1, \ldots, r_{12}$ coming from the matrix equations

$$r_1 T_1 + \ldots + r_{12} T_{12} = T$$

$$r_1 S_1 + \ldots + r_{12} S_{12} = S.$$

Appendix 2

Coefficient Matrix for the Rigid Embeddings of Cu

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 1 & -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 1 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

The rank of this matrix is 23.